Review & Complement

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Announcement

- 4/20 (下下週三) 期中考
  - 範圍: 到第四章 (有*章節不考)
- 下週三、週四複習
  - 把想問的問題、想講解的習題、想講解的章節在下週二午夜前讓老師知道
- 勾選習題
  - http://speech.ee.ntu.edu.tw/~tlkagk/courses/LA_2016/Lecture/problem.pdf

More about Rank

Review: Rank A

- Maximum number of Independent Columns
- Number of Pivot Columns
- Number of Non-zero rows
- Number of Basic Variables
- Dim (Col A) = Dim (Row A) = Dim (Col A^T)
- Dimension of the range of function A

Properties of Rank

- A is a m x n matrix.

$$Rank\ \mathrm{A} \ \leq \min(\, m \,,\, n \,)$$

  - A is said to have **full rank** if Rank A = m or Rank A = n.
  - A is said to be **rank deficient** if it does not have full rank.
- Rank A = Rank A^T

Properties of Rank

- If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq min\big(Rank(A), Rank(B)\big)$$

- If B is a matrix of rank n, then

$$Rank(AB) = Rank(A)$$

- If A is a matrix of rank n, then

$$Rank(AB) = Rank(B)$$

Properties of Rank

- If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq Rank(A)$$

- If A is a m x n matrix, and Q is a m x m invertible matrix.

$$Rank(QA) = Rank(A)$$

Invertible matrix is a product of elementary matrices.

Elementary row operation will not change the row space

$\Rightarrow$ dim of row space $\Rightarrow$ Rank

Properties of Rank

- If A is a m x n matrix, and B is a n x k matrix.

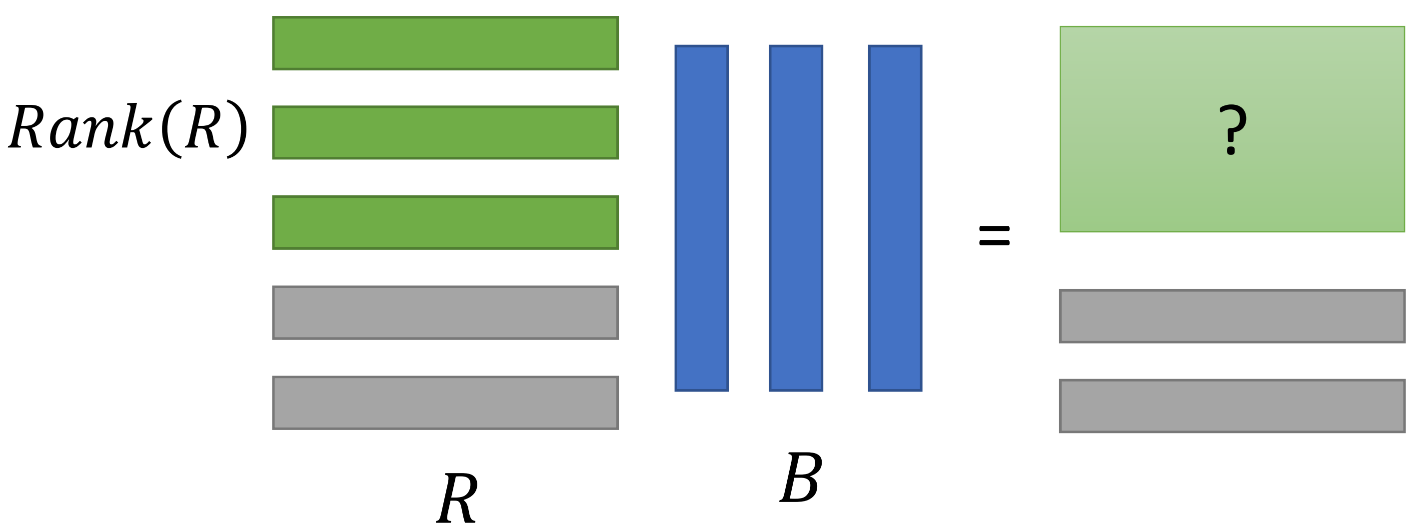
$$Rank(AB) \leq Rank(A)$$

$PA = R$

P is an invertible matrix

$$Rank(AB) = Rank(PAB) = Rank(RB)$$

$$Rank(A) = Rank(PA) = Rank(R)$$



It can also have zero rows

Properties of Rank

- If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \le Rank(A) \Rightarrow Rank(AB) \le Rank(B)$$

- If B is a matrix of rank n, then $Rank(AB) = Rank(A)$
- If A is a matrix of rank n, then $Rank(AB) = Rank(B)$

$$Rank(AB) = Rank(B^T A^T) \le Rank(B^T) = Rank(B)$$

$$\mathrm{Rank}(A^T) = n = \mathrm{Rank}(A)$$

Theorem 4.9 (P258)

- If V and W are subspaces of R^n with V contained in W, then dim V $\leq$ dim W
- If dim V = dim W, V=W
- Proof:

B_V is a basis of V, V in W, B_V in W

$\Rightarrow$ B_V is an independent set in W

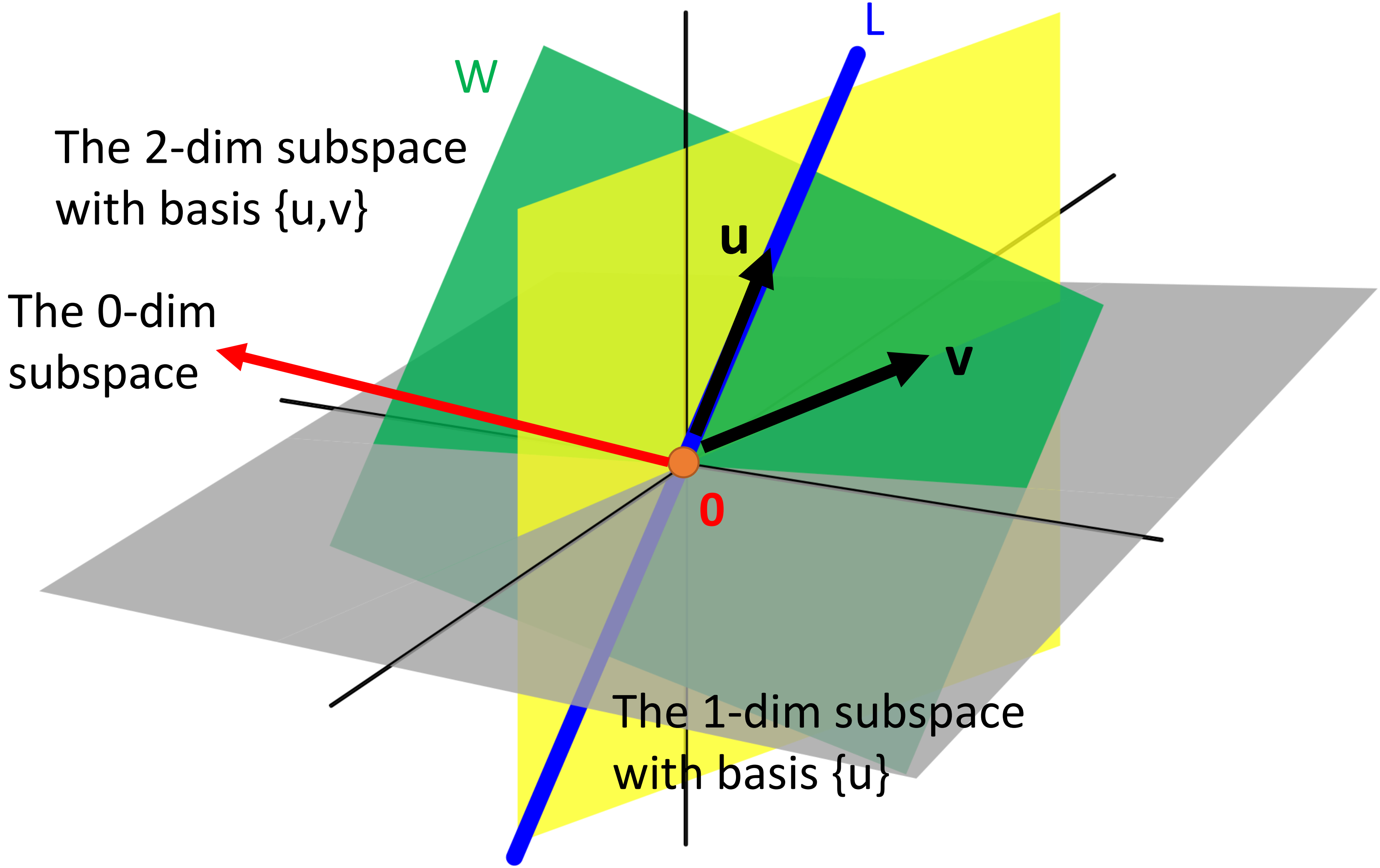
By extension theorem, B_V is in the basis of W $\Rightarrow$ dim V $\leq$ dim W

If dim V = dim W =k

B_V is a linear independent set in W, with k elements

$\Rightarrow$ It is also the span of W

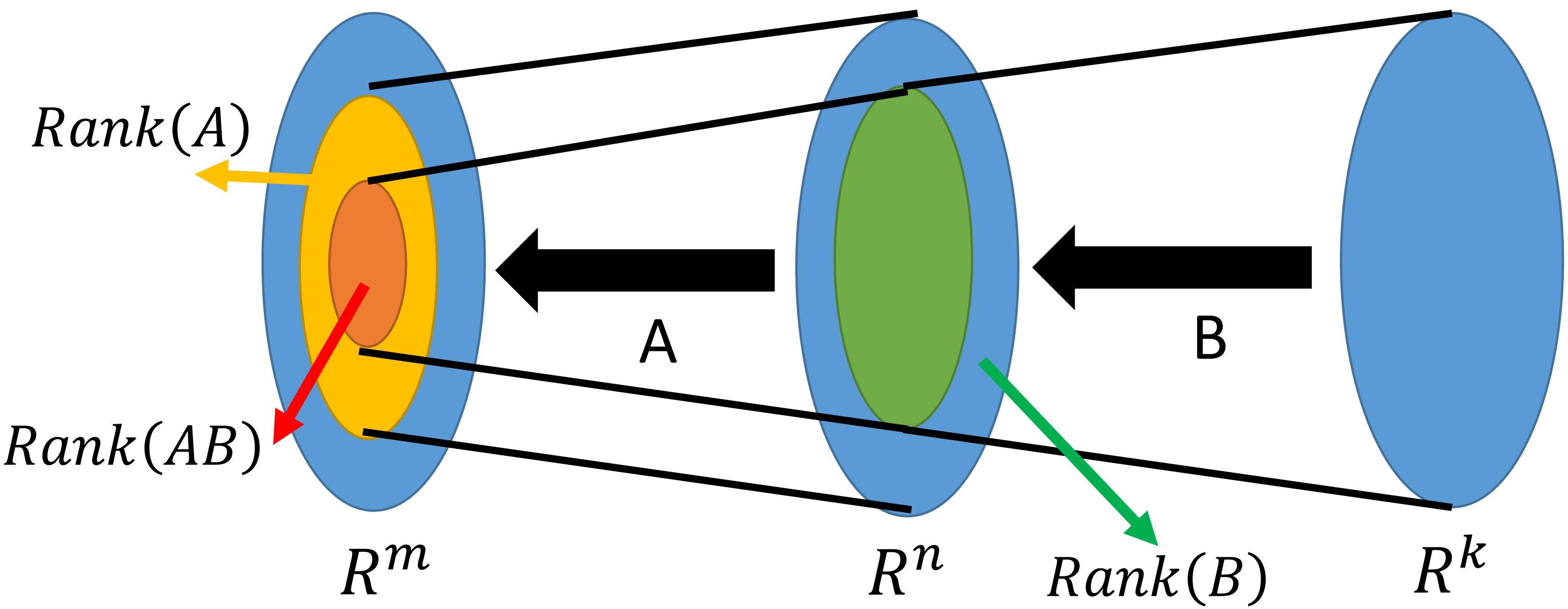
R^3 is the only 3-dim subspace of itself



Properties of Rank

- If A is a m x n matrix, and B is a n x k matrix.

$$Rank(AB) \leq Rank(A)$$



HW: Proof $Rank(A+B) \leq Rank(A) + Rank(B)$

Properties of Rank

$$Rank(AB) \le min\big(Rank(A), Rank(B)\big)$$

Dependent

Independent

More about Determinants

Properties of Determinants

- Basic Property 1: $det(I) = 1$
- Basic Property 2: Exchange rows reverse the sign of det
  - If a matrix A has 2 equal rows, det A = 0
- Basic Property 3: Determinant is "linear" for each row
  - A row of zeros, det A = 0

$$det\left(\begin{bmatrix} ta & tb \\ c & d \end{bmatrix}\right) = t\,det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

$$det\left(\begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix}\right) = det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + det\left(\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}\right)$$

A is invertible $\Longleftrightarrow$ det(A) ≠ 0

Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is not invertible:

A is not invertible $\Rightarrow$ AB is not invertible

$\Rightarrow$ det AB = 0

A is not invertible $\Rightarrow$ det A = 0

$\Rightarrow$ det A det B = 0

Properties of Determinants

- $det(AB) = det(A)det(B)$
- Proof:

If A is invertible:

$$A = E_k \cdots E_2 E_1$$

You have to proof that det EA = det E det A

(E is elementary matrix)

$$det(A) = det(E_k) \cdots det(E_2)det(E_1)$$

$$\begin{aligned} det(A)det(B) &= det(E_k) \cdots det(E_2)det(E_1)det(B) \\ &= det(E_k) \cdots det(E_2)det(E_1 B) \\ &= det(E_k \cdots E_2 E_1 B) = det(AB) \end{aligned}$$

Properties of Determinants

det E = det E^T …… in the textbook

- det A = det A^T
- Proof: $detA = \sum \pm n!\, terms$

Format of each term: $a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\omega}$

Sorted by column indices

Find an element in each row

permutation of 1,2, …, n

Format of each term: $a_{\alpha'1}a_{\beta'2}a_{\gamma'3}\cdots a_{\omega'n}$

Find an element in each column

permutation of 1,2, …, n

A v.s. A^T

- Rank A = Rank A^T
- det A = det A^T

$$A = \begin{matrix} \text{趙} & \text{靈} & \text{兒} \\ \text{林} & \text{月} & \text{如} \\ \text{阿} & \text{奴} & \end{matrix} \quad \Rightarrow \quad A^T = \begin{matrix} \text{趙} & \text{林} & \text{阿} \\ \text{靈} & \text{月} & \text{奴} \\ \text{兒} & \text{如} & \end{matrix}$$

Name A = Name A^T

Dependent and Independent Set

Properties (P81)

- For vector sets with one vector $\{\boldsymbol{u}\}$:

$\boldsymbol{u} \neq \mathbf{0}$ dependent $\boldsymbol{u} = \mathbf{0}$ independent

- For vector sets with two vector $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$:

$\boldsymbol{u}_1 = \mathbf{0}$ or $\boldsymbol{u}_2 = \mathbf{0}$ $\boldsymbol{u}_2$ is a multiple of $\boldsymbol{u}_1$

dependent dependent

- For a vector set with three vector $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1 + \boldsymbol{e}_2\}$

dependent

Properties (P81)

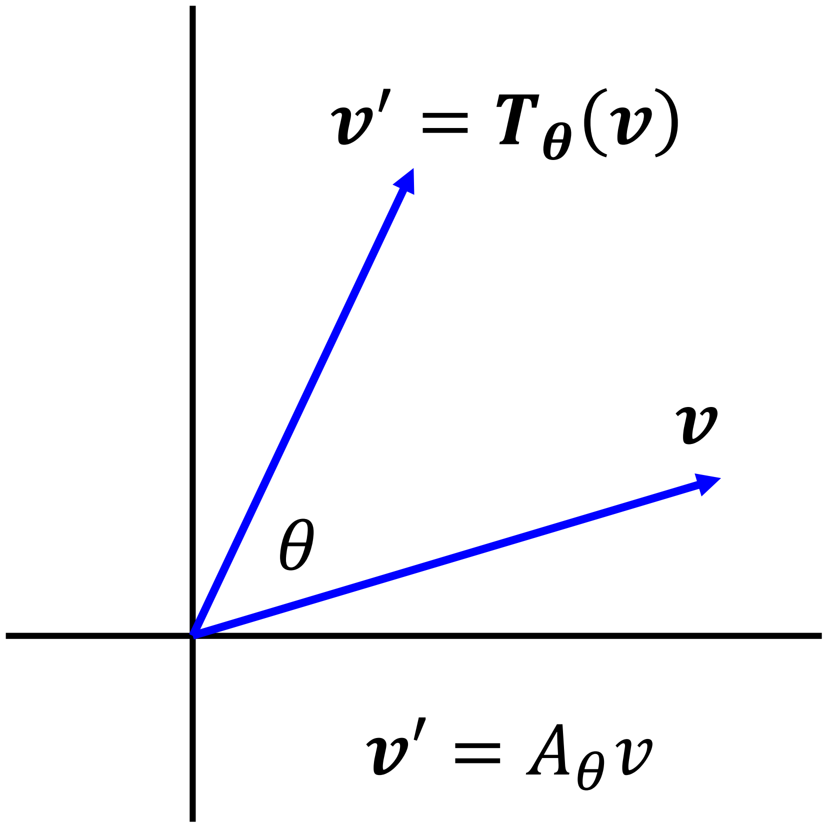
- Let $\{u_1, u_2, \cdots, u_k\}$ is independent

  v is not in Span $\{u_1, u_2, \cdots, u_k\}$ $\Leftrightarrow$ $\{u_1, u_2, \cdots, u_k, v\}$ is independent

- Every vector set of R^n containing more than n vectors must be dependent.
- If no vector can be removed from vector set S without changing its span, S is independent.
- Theorem 1.9 (yourself)

Rotation Matrix

Rotation Matrix



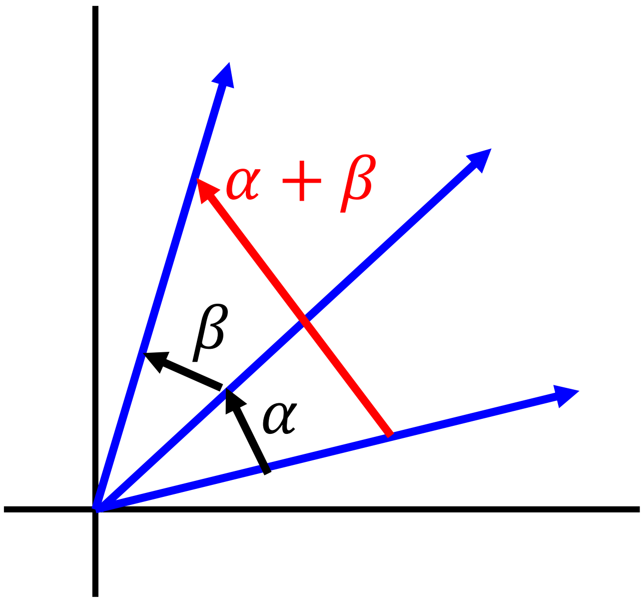
$$\boldsymbol{T_\theta(e_1)} = \begin{bmatrix} \boldsymbol{cos\theta} \\ \boldsymbol{sin\theta} \end{bmatrix}$$

$$\boldsymbol{T_\theta(e_2)} = \begin{bmatrix} \boldsymbol{-sin\theta} \\ \boldsymbol{cos\theta} \end{bmatrix}$$

$$A_\theta = \begin{bmatrix} \boldsymbol{cos\theta} & \boldsymbol{-sin\theta} \\ \boldsymbol{sin\theta} & \boldsymbol{cos\theta} \end{bmatrix}$$

$$A_{0^\circ} = I_2$$

Rotation Matrix



$$A_{\alpha+\beta} = A_\alpha A_\beta = A_\beta A_\alpha$$

$$A_\alpha = \begin{bmatrix} \boldsymbol{cos}\alpha & -\boldsymbol{sin}\alpha \\ \boldsymbol{sin}\alpha & \boldsymbol{cos}\alpha \end{bmatrix}$$

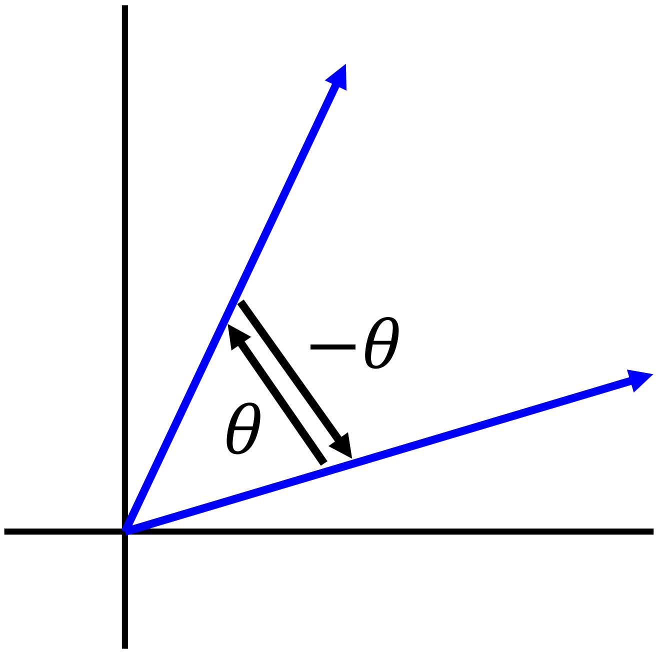
$$A_\beta = \begin{bmatrix} \boldsymbol{cos}\beta & -\boldsymbol{sin}\beta \\ \boldsymbol{sin}\beta & \boldsymbol{cos}\beta \end{bmatrix}$$

$$A_\alpha A_\beta = \begin{bmatrix} \boldsymbol{cos}\alpha\boldsymbol{cos}\beta - \boldsymbol{sin}\alpha\boldsymbol{sin}\beta & -\boldsymbol{cos}\alpha\boldsymbol{sin}\beta - \boldsymbol{sin}\alpha\boldsymbol{cos}\beta \\ \boldsymbol{sin}\alpha\boldsymbol{cos}\beta + \boldsymbol{cos}\alpha\boldsymbol{sin}\beta & -\boldsymbol{sin}\alpha\boldsymbol{sin}\beta + \boldsymbol{cos}\alpha\boldsymbol{cos}\beta \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{cos}(\alpha + \beta) & -\boldsymbol{sin}(\alpha + \beta) \\ \boldsymbol{sin}(\alpha + \beta) & \boldsymbol{cos}(\alpha + \beta) \end{bmatrix} = A_{\alpha+\beta}$$

Rotation Matrix

$$(A_\theta)^{-1} = A_{-\theta}$$



$$A_\theta A_{-\theta} = A_{\theta-\theta}$$

$$= A_{0^\circ} \quad = I_2$$

$$A_\theta = \begin{bmatrix} \boldsymbol{cos\theta} & \boldsymbol{-sin\theta} \\ \boldsymbol{sin\theta} & \boldsymbol{cos\theta} \end{bmatrix}$$

$$A_{-\theta} = \begin{bmatrix} \boldsymbol{cos(-\theta)} & \boldsymbol{-sin(-\theta)} \\ \boldsymbol{sin(-\theta)} & \boldsymbol{cos(-\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{cos\theta} & \boldsymbol{sin\theta} \\ \boldsymbol{-sin\theta} & \boldsymbol{cos\theta} \end{bmatrix}$$